

Hermitian modular forms congruent to 1 modulo p .

Michael Hentschel

Lehrstuhl A für Mathematik, RWTH Aachen University, 52056 Aachen, Germany,
hentschel@mathA.rwth-aachen.de

Gabriele Nebe

Lehrstuhl D für Mathematik, RWTH Aachen University, 52056 Aachen, Germany,
nebe@math.rwth-aachen.de

Abstract

For any natural number ℓ and any prime $p \equiv 1 \pmod{4}$ not dividing ℓ there is a Hermitian modular form of arbitrary genus n over $L := \mathbb{Q}[\sqrt{-\ell}]$ that is congruent to 1 modulo p which is a Hermitian theta series of an O_L -lattice of rank $p-1$ admitting a fixed point free automorphism of order p . It is shown that also for non-free lattices such theta series are modular forms.

1 Introduction.

The purpose of the present note is to generalize the construction of Siegel modular forms that are congruent to 1 modulo a suitable prime p given in [3] to the case of Hermitian modular forms over $L := \mathbb{Q}[\sqrt{-\ell}]$. For $\ell = 1$ and $\ell = 3$ this was done in [11], in fact we use the same strategy by constructing an even unimodular lattice Λ as an ideal lattice in $K := L[\zeta_p]$ for any prime $p \equiv 1 \pmod{4}$ not dividing ℓ . The existence of Λ essentially follows from class field theory and is predicted by [2, Théorème 2.3, Proposition 3.1 (1)] (see also [1, Corollary 2]). Since the ring of integers O_L is in general not a principal ideal domain the lattice Λ is not necessarily a free O_L -module. We are not aware of an explicit statement in the literature that the genus n Hermitian theta series $\theta^{(n)}(\Lambda)$ of such a lattice Λ is a modular form for the full modular group. Therefore the first section sketches a proof. In fact the proofs in the literature never seriously use the fact that the lattice is a free O_L -module. The next section applies the results of [2] and [1] to the special case of the field $K = \mathbb{Q}[\sqrt{-\ell}, \zeta_p]$ and proves the existence of a Hermitian O_K -lattice Λ_h that is an even unimodular \mathbb{Z} -lattice (with respect to the trace of the Hermitian form). The invariance under O_K yields both, a Hermitian O_L -module structure on Λ_h and an O_L -linear automorphism (the multiplication by the primitive p -th root of unity $\zeta_p \in O_K$) of order p acting fixed point freely on $\Lambda_h \setminus \{0\}$. Therefore all but the first coefficient in $\theta^{(n)}(\Lambda_h)$ are multiples of p yielding the desired Hermitian modular form.

2 Hermitian theta-series are Hermitian modular forms.

Let $\ell \in \mathbb{N}$ such that $-\ell$ is a fundamental discriminant (which means that either $\ell \equiv -1 \pmod{4}$ is square-free or $\ell = 4m$, where $m \equiv 2$ or $1 \pmod{4}$ is square-free). Let $L :=$

$\mathbb{Q}[\sqrt{-\ell}]$ be the imaginary quadratic number field of discriminant $-\ell$, with ring of integers O_L and inverse different

$$O_L^* := \{a \in L \mid \text{Tr}_{L/\mathbb{Q}}(aO_L) \subset \mathbb{Z}\} = \sqrt{-\ell}^{-1} O_L.$$

Let (V, h) be a finite dimensional positive definite Hermitian vector space over L . This section extends the results in [4] to not necessarily free Hermitian O_L -lattices in (V, h) . Note that we use a different scaling for the Hermitian form resulting in the additional factor of 2 in the definition of the Hermitian Siegel theta series below. It is already stated in [4] that the authors restrict to free lattices “for convenience” and that the same results hold in the more general context. The **full modular group**

$$\Gamma_n := \left\langle \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mid B \in O_L^{n \times n} \text{ Hermitian}, U \in \text{GL}_n(O_L) \right\rangle$$

(see [7], [5, Anhang V], [8]) for the proof that these matrices really generate) acts on the Hermitian half space by

$$Z \mapsto Z + B, \quad Z \mapsto \overline{U}^t Z U, \quad Z \mapsto -Z^{-1}$$

for the respective generators.

Theorem 1. *Let $\Lambda_h \leq (V, h)$ be an O_L -lattice such that the O_L -dual lattice*

$$\Lambda_h^* := \{v \in V \mid h(v, \Lambda_h) \subset O_L\} = \sqrt{-\ell} \Lambda_h = (O_L^*)^{-1} \Lambda_h.$$

Then its Hermitian theta series

$$\theta^{(n)}(\Lambda_h)(Z) := \sum_{(x_1, \dots, x_n) \in \Lambda_h^n} \exp(2\pi i \text{trace}(h(x_i, x_j)Z))$$

is a Hermitian modular form for the full modular group Γ_n . Here $\text{trace} : L^{n \times n} \rightarrow \mathbb{Q}$ denotes the composition of the matrix trace with the trace of L over \mathbb{Q} .

Proof. For $x := (x_1, \dots, x_n) \in \Lambda_h^n$ the Hermitian matrix $H := H_x := (h(x_i, x_j)) \in (O_L^*)^{n \times n}$, so for any Hermitian matrix $B \in O_L^{n \times n}$ the trace $\text{trace}(HB)$ is in \mathbb{Z} . This shows the invariance of $\theta^{(n)}(\Lambda_h)$ under $Z \mapsto Z + B$. Similarly

$$\text{trace}(H_x \overline{U}^t Z U) = \text{trace}(U H_x \overline{U}^t Z) = \text{trace}(H_{xU} Z)$$

so the transformation $Z \mapsto \overline{U}^t Z U$ for $U \in \text{GL}_n(O_L)$ just changes the order of summation in $\theta^{(n)}(\Lambda_h)$. It remains to prove the theta-transformation formula

$$(\star) \quad \theta^{(n)}(\Lambda_h)(-Z^{-1}) = \det(Z/i)^d \theta^{(n)}(\Lambda_h)(Z)$$

also for non-free O_L -lattices Λ_h of dimension d that satisfy $\Lambda_h^* = (O_L^*)^{-1} \Lambda_h$. But Poisson summation only depends on the abelian group structure, not on the underlying module, so the usual proof (see for instance [9, p. 111]) can be adopted to the situation here (for details we refer to [6]): Using the Identity Theorem, it suffices to prove (\star) for $Z = iY$, Y Hermitian

positive definite. Let $\varphi : \mathbb{R}^{2dn} \rightarrow \mathbb{C}^{d \times n}$ be the obvious isomorphism and consider Λ_h^n as a lattice $\tilde{\Lambda}$ in $\mathbb{C}^{d \times n}$ choosing coordinates with respect to an orthonormal basis of (\mathbb{C}^d, h) . Then there is some $F \in \mathbb{R}^{2dn \times 2dn}$ such that $\tilde{\Lambda} = \varphi(F\mathbb{Z}^{2dn})$ and $H_{\varphi(x)} = \overline{\varphi(Fx)}^{tr} \varphi(Fx)$. Then

$$\theta^{(n)}(\Lambda_h)(iY) = \sum_{g \in \mathbb{Z}^{2dn}} \psi(g)$$

where

$$\psi : \mathbb{R}^{2dn} \rightarrow \mathbb{C}, x \mapsto \exp(-2\pi \operatorname{trace}(\overline{\varphi(Fx)}^{tr} \varphi(Fx)Y)).$$

The condition $\Lambda_h^* = (O_L^*)^{-1} \Lambda_h$ implies that $|\det(F)| = 1$ and we can apply the usual Poisson summation to get the result as in [9, pp. 110-112]. \square

3 Congruences of Hermitian theta-series.

Let p be a prime $p \equiv 1 \pmod{4}$ such that ℓ is not a multiple of p . This section constructs a Hermitian O_L -lattice (Λ, h) of rank $p-1$ admitting an automorphism of order p such that the \mathbb{Z} -lattice $(\Lambda, \operatorname{Tr}_{L/\mathbb{Q}}(h))$ is a positive definite even unimodular lattice. The existence of such a lattice follows from the much more general result [2, Théorème 2.3] together with [2, Proposition 3.1] which are based on Artin's reciprocity law in global class field theory (see [10, Theorem (V.3.5)]). For our special case it is however more convenient to use [1, Corollary 2], which is essentially a consequence of [2, Théorème 2.3].

To this aim we consider the number field $K = \mathbb{Q}[\sqrt{-\ell}][\zeta_p] = LM$ with $M = \mathbb{Q}[\zeta_p]$, where $\zeta_p = \exp(\frac{2\pi i}{p})$ is a primitive p -th root of unity. Then K is an abelian number field of degree $2(p-1)$ over \mathbb{Q} which is a multiple of 8. The field K is totally complex and admits an involution $\bar{}$, the complex conjugation, with fixed field F the totally real subfield of K .

The following lemma is well known.

Lemma 2. *K/F is unramified at all finite primes.*

Proof. The discriminant $d_{K/F}$ of K/F divides the discriminant of any F -basis of K that consists of integral elements. For $B_1 = (1, \sqrt{-\ell})$ one finds $d_{B_1} = \det(\operatorname{Tr}_{K/F}(b_i b_j)) = -4\ell$ and for $B_2 = (1, \zeta_p)$ one get $d_{B_2} = \zeta_p^{-2}(\zeta_p^2 - 1)^2$ which generates an ideal of norm p^2 in F . Since p is an odd prime not dividing ℓ , the gcd of these two discriminants is 1 and hence $d_{K/F} = 1$ which implies the lemma. \square

Since all real embeddings of F extend to complex embeddings of K and $[K : \mathbb{Q}] = 2(p-1) \equiv 0 \pmod{8}$ [1, Corollary 2] yields the existence of a fractional O_K -ideal \mathcal{A} in K and a totally positive element $d \in F$ such that the O_K -module \mathcal{A} together with the symmetric integral bilinear form

$$b_d : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{Z}, (x, y) \mapsto \operatorname{trace}_{K/\mathbb{Q}}(dx\bar{y})$$

is an even unimodular \mathbb{Z} -lattice $\Lambda := (\mathcal{A}, b_d)$. This means that $b_d(x, x) \in 2\mathbb{Z}$ for all $x \in \mathcal{A}$ and

$$\Lambda^\# := \{x \in K \mid b_d(x, y) \in \mathbb{Z} \text{ for all } y \in \mathcal{A}\} = \Lambda.$$

Corollary 3. *The O_L -lattice $\Lambda_h := (\mathcal{A}, h(x, y) := \text{Tr}_{K/L}(dx\bar{y}))$ is a Hermitian O_L -lattice with automorphism $x \mapsto \zeta_p x$ of order p such that $\Lambda_h^* = (O_L^*)^{-1} \Lambda_h$.*

Proof. Since \mathcal{A} is an ideal of K , the multiplication by $\zeta_p \in O_K$ preserves the lattice \mathcal{A} . It also respects the Hermitian form h , because

$$h(\zeta_p x, \zeta_p y) = \text{Tr}_{K/L}(d\zeta_p x \overline{\zeta_p y}) = \text{Tr}_{K/L}(d\zeta_p \zeta_p^{-1} x \bar{y}) = h(x, y).$$

The fact that $\Lambda_h^* = (O_L^*)^{-1} \Lambda_h$ follows from the unimodularity of the integral lattice Λ : For $y \in K$ we obtain

$$b_d(x, y) = \text{trace}_{L/\mathbb{Q}}(h(x, y)) \in \mathbb{Z} \text{ for all } x \in \mathcal{A} \Leftrightarrow h(x, y) \in O_L^* \text{ for all } x \in \mathcal{A}$$

using the fact that \mathcal{A} is an O_L -module and h is Hermitian over O_L . Hence $\Lambda_h^* = (O_L^*)^{-1} \Lambda_h^\# = (O_L^*)^{-1} \Lambda_h$. \square

Together this implies the existence of a Hermitian modular form of weight $p - 1$ that is congruent to 1 modulo p for more general imaginary quadratic number fields than those treated in [11]:

Theorem 4. *Let $L = \mathbb{Q}[\sqrt{-\ell}]$ be an imaginary quadratic number field ($-\ell$ a fundamental discriminant) and let p be a prime $p \equiv 1 \pmod{4}$ not dividing ℓ . Then for arbitrary genus $n \geq 1$ there is a Hermitian modular form*

$$F_{p-1}^{(n)} \in M_{p-1}(SU_n(O_L))$$

for the full modular group over the ring of integers O_L of L such that

$$F_{p-1}^{(n)} \equiv 1 \pmod{p}.$$

Proof. Corollary 3 constructs a Hermitian O_L -lattice Λ_h of rank $p - 1$ admitting an automorphism of order p (which necessarily acts fixed point freely) such that $\Lambda_h^* = (O_L^*)^{-1} \Lambda_h$. By Theorem 1 its Siegel theta series is a Hermitian modular form for the full modular group. Since Λ_h admits a fixed point free automorphism of order p , all the representation numbers

$$R_A := |\{(x_1, \dots, x_n) \in \Lambda_h^n \mid (h(x_i, x_j)) = A\}|$$

for any non-zero Hermitian matrix $A \in L^{n \times n}$ are multiples of p and hence

$$F_{p-1}^{(n)} := \theta_{\Lambda_h}^{(n)} \equiv 1 \pmod{p}$$

provides the desired Hermitian modular form. \square

Since the root lattice E_8 is the unique even unimodular lattice of dimension 8, we obtain the following corollary.

Corollary 5. *Let $\ell \in \mathbb{N}$ be not a multiple of 5. Then the root lattice E_8 has a Hermitian structure as a lattice Λ_h over the ring of integers of $\mathbb{Q}[\sqrt{-\ell}]$ such that $\text{Aut}(\Lambda_h)$ contains an element of order 5.*

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